PART II

"*While algebra and analysis provide the foundations of mathematics, geometry is at the core*"[1]

S.-S. Chern

[1] Chern, S.-S. Introduction. As cited in: F. Dillen & L. Verstraelen (eds.), *Handbook of Differential Geometry*, Vol. 1. Elsevier, Amsterdam, Netherlands, 2000.

Chapter 9. From Lamé Curves to Gielis Transformations

9.1 Riemann's Geometric Ideas

In the mid-19th century geometry went multidimensional when Riemann defined analytically n-fold extended manifolds, where each point can be described by an n-tuple of numbers (their coordinates). Following Gauss' work on surfaces embedded in E^3 he also used systems of local coordinates, as a relevant extension of the geometry of surfaces in Euclidean space E^3 based on the Pythagorean theorem [61]. In his famous memoir, Riemann explicitly mentioned the geometric tangent $2-D$ indicatrix of fourth order $x^4 + y^4 = 1$ as an extension of the Euclidean circle $x^2 + y^2 = 1$. He thus conceived of distance metrics (9.1) [47]:

$$ds = \left\{ \sum_{i=1}^{n} (dx_i)^p \right\}^{1/p} \tag{9.1}$$

With the Euclidean distance for $p = 2$, this forms a bridge between classical Euclidean geometry and Riemannian geometry (with a "quadratic" distance form based on Pythagoras) on the one hand, and Finsler geometry ("without the quadratic restriction") and metric spaces ("with the m-th root metric") on the other [30]. In the infinitesimal form (9.1) the simplest so-called Riemann-Finsler geometries are defined [47]. These also find applications beyond geometry. For example, in the early 1990s P.L. Antonelli proposed a Lamé metric in ecology [2]:

$$ds = e^{\varphi(x)}[(dx_1)^n + (dx_2)^n]^{1/n} \tag{9.2}$$

Equation (9.1) can account for any number of dimensions, for example, defining Minkowski distances in L_p spaces [83]:

$$||x||_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p} \tag{9.3}$$

These types of equations are systematically discussed for the first time in a booklet *Examen des Différentes Méthodes Employées pour Résoudre les Problèmes de Géométrie*, published in 1818 by Gabriel Lamé [66]:

$$x^n + y^n = R^n; \qquad x^n + y^n = 1 \tag{9.4}$$

These equations describe the so-called one-parameter Lamé curves (Figure 19). Since a circle is defined in a plane by a total of three numbers (the coordinates of its center and its radius), the totality of all circles in the plane is a 3-dimensional manifold. The totality of all ellipses in the plane is a 5-dimensional manifold with the major and minor axes and orientation of the ellipses [61].

However, all Lamé curves defined by (9.4), which includes all circles, all squares (the inscribed squares for $n = 1$ and the circumscribing squares for $n \to \infty$), all astroids (for $n = 2/3$), as well as all supercircles for any $n > 0$ (Equation 9.5a), still constitute only a 4-dimensional manifold, or a 5-dimensional manifold for superellipses (Equation 9.5b). Gielis curves are a relevant extension of Lamé curves adding a few more parameters. These curves and transformations provide for a unified description of natural shapes [47]. In Part IV many examples are shown.

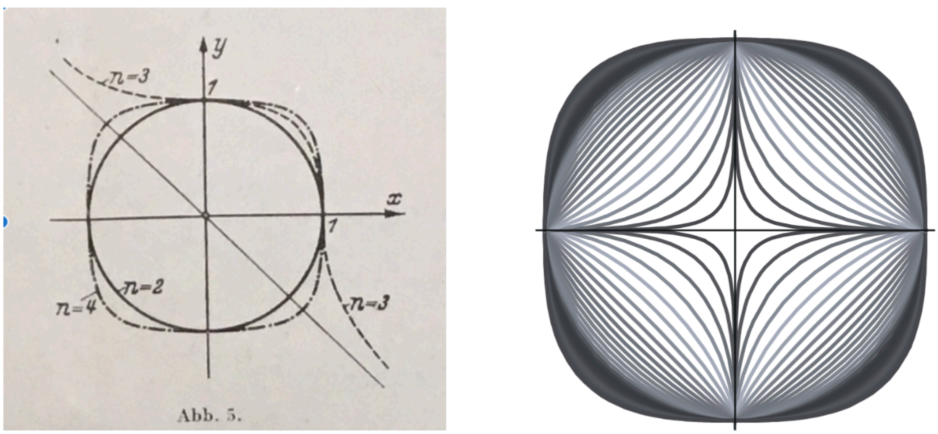

Figure 19. Left: Lamé curves for odd and even exponent values [53]. Right: supercircles, also for $n < 2$.

9.2 Lamé Curves and Superellipses

In a Cartesian (x, y) system, Equation (9.4) (with n a positive integer which Lamé assumed > 1) defines the so-called Lamé curves with base radius R. For even n, the curve (9.4) is a closed curve without real double and inflection points and with the four symmetry axes $x = 0$, $y = 0$, $x = \pm y$. For odd n, it is a single curve without real double point, with the two inflection points $(1; 0)$ and $(0; 1)$, the symmetry axis $x = y$ and the asymptote $x = -y$ (Figure 19) [53]. Supercircles or superellipses, both a subset of Lamé curves and a generalization of the circle, are based on Equations (9.5a) and (9.5b) with n a positive integer and using absolute values to ensure that shapes are closed.

$$|x|^n + |y|^n = R^n \tag{9.5a}$$

$$\left|\frac{x}{A}\right|^n + \left|\frac{y}{B}\right|^n = 1 \tag{9.5b}$$

In fact, many problems of analytic geometry that have become part of modern geometric techniques and textbooks were first solved by Gabriel Lamé. Here we find the first mention of the equation of a plane in the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Lamé's intention was to open geometry to the study of crystals, but it was only around 1993 that his equation was also applied to model square bamboos (Figure 20 left).

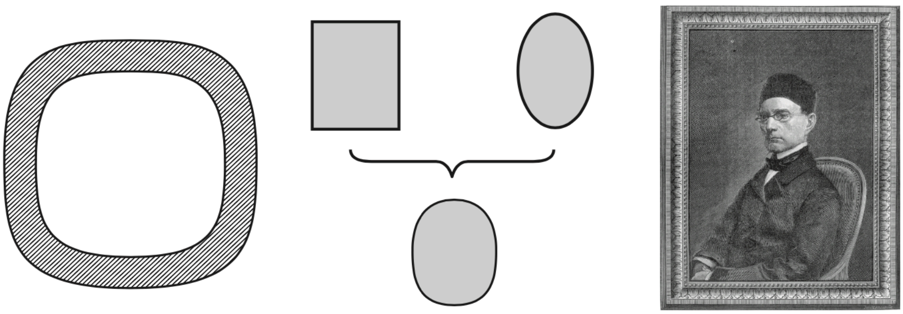

Figure 20. Left: superelliptical section of a square bamboo. Center: optimal solution according to Piet Hein. Right: Gabriel Lamé [41].

Gabriel Lamé (1795-1870) attended the École Polytechnique in Paris from 1813 to 1818 and graduated from the School of Mines in 1820. Over the next decade, from 1820 to 1831, Lamé worked in Saint-Petersburg responsible for railways and bridges [51]. After returning to Paris, Lamé became Professor of Physics at the École Polytechnique. Lamé's mathematical discoveries are closely linked to his research on the theory of elasticity. He is considered a father of mathematical physics with the introduction of the parameters (invariants) of a scalar field of first and second kind [41].

$$\Delta_1 F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2} \tag{9.6a}$$

$$\Delta_2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} \tag{9.6b}$$

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \vartheta^2} \tag{9.6c}$$

$\Delta_2 F$ and Δu are the Laplacian, expressed in Cartesian and polar coordinates respectively. The curvilinear coordinates and the differential parameters introduced by Lamé inspired the Italian school of differential geometry with Ricci, Levi-Civita and Beltrami. Gabriel Lamé should not only be considered as one of the founders of differential geometry, but also of Riemannian geometry in the opinion of Elie Cartan (1869-1951) who was a leading geometer of the 20th century [55]. He was the first to apply curvilinear coordinates in space using an orthogonal system, giving the length of an element as:

$$ds^2 = H^2 d\rho^2 + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$$

He was naturally led to Fermat's Last Theorem since this is exactly Equation (9.5a) and he proved the case for $n = 7$. The recurrence formula that gives rise to Fibonacci numbers was used by Lamé to develop the Euclidean algorithm, to determine the greatest common divisor of two integers [69], and is still in use today. Gaston Darboux spoke of the immortal works of Gabriel Lamé and Gauss called him the best French mathematician of his time. On the Eiffel Tower, the names of Fourier, Carnot and Lamé are very close together, all on the Bourdonnais side of the Tower.

Due to the work of the Danish mathematician and inventor Piet Hein, Lamé curves became very popular in the 1960s, in objets d'art, furniture, pottery, fabric patterns, etc. But his major achievement to date is a sunken oval shopping plaza, promenade and pool in the center of Stockholm. For Piet Hein the superellipse was an iconic solution between a round and a square worldview: "*Man is the animal that draws lines, which he himself then stumbles over. In the whole pattern of civilization there have been two tendencies, one toward straight lines and rectangular patterns and one toward circular lines. There are reasons, mechanical and psychological, for both tendencies. Things made with straight lines fit well together and save space. And we can move easily - physically or mentally - around things made with round lines. But we are in a straitjacket, having to accept one or the other, when often some intermediate form would be better. To draw something freehand - such as the patchwork traffic circle they tried in Stockholm - will not do. It isn't fixed, it isn't definite like a circle or square. You don't know what it is. It is not esthetically satisfying. The superellipse solved the problem. It is neither round nor rectangular, but in between. Yet it is fixed, it is definite - it has a unity. The superellipse has the same beautiful unity as the circle and ellipse but is less obvious and less plain. The superellipse frees us from the straitjacket of simple curves such as lines and planes.*" [42]

9.3 Trigonometry of Supercircles and a Generalization of π

The sector of area bounded by the curve between the positive x-axis and the driving ray from the zero point to the point of the curve (x, y) (Figure 21) is denoted by $(1/2)\nu$, because in the case of $n = 2$ of the circle it is half the central angle of the sector. Its double value is [53]:

$$\nu = \int_0^y x dy + \int_x^1 y dx = \int_{(x=1,y=0)}^{x,y} (x dy - y dx) \tag{9.7}$$

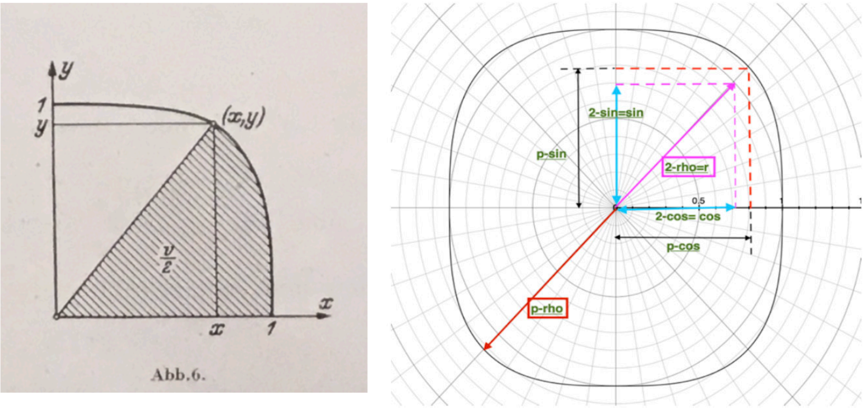

Figure 21. Left: sector of superellipse [53]. Right: trigonometric functions on a supercircle.

Depending on whether the expression is of y in x, or of x in y according to (9.5b), the first or the second of the following integrals is obtained:

$$\nu = \int_x^1 \frac{dx}{(1-x^n)^{\frac{n-1}{n}}} \quad \text{or} \quad \nu = \int_0^y \frac{dy}{(1-y^n)^{\frac{n-1}{n}}} \tag{9.8}$$

These integrals are denoted by $\arccos_n(x)$ and $\arcsin_n(y)$ and their inverse functions by $\cos_n \nu$ and $\sin_n \nu$, thus:

$$\nu = \arccos_n(x) = \int_x^1 \frac{dx}{(1-x^n)^{\frac{n-1}{n}}}, \quad x = \cos_n \nu \tag{9.9a}$$

$$\nu = \arcsin_n(x) = \int_0^y \frac{dy}{(1-y^n)^{\frac{n-1}{n}}}, \quad y = \sin_n \nu \tag{9.9b}$$

These integrals pass into the functions $\arccos(x)$ and $\arcsin(y)$ for $n = 2$. The functions $\tan_n \nu$ and $\mathrm{cotan}_n \nu$ are defined by:

$$\tan_n \nu = \frac{\sin_n \nu}{\cos_n \nu}, \quad \mathrm{cotan}_n \nu = \frac{\cos_n \nu}{\sin_n \nu} \tag{9.10}$$

This leads to the natural generalization in the system $\dot{x} = y^p$, $\dot{y} = -x^p$, of the differential equation system $\dot{x} = y$, $\dot{y} = -x$. Combining (9.4) right (the unit circle) and (9.9a)-(9.9b) it follows that [53]:

$$\frac{dx}{d\nu} = -(1-x^n)^{\frac{n-1}{n}} = -y^{n-1}\,, \quad \frac{dy}{d\nu} = (1-y^n)^{\frac{n-1}{n}} = x^{n-1} \tag{9.11}$$

These are the differential equations for the functions $\sin_n \nu$ and $\cos_n \nu$, namely:

$$\frac{d}{d\nu}\cos_n \nu = -\sin_n^{n-1} \nu\,, \quad \frac{d}{d\nu}\sin_n \nu = \cos_n^{n-1} \nu \tag{9.12}$$

Furthermore, it follows according to (9.12):

$$\begin{aligned} \frac{d^2}{d\nu^2}\cos_n \nu &= -(n-1)\cos_n^{n-1} \nu \sin_n^{n-2} \nu \\ \frac{d^2}{d\nu^2}\sin_n \nu &= -(n-1)\sin_n^{n-1} \nu \cos_n^{n-2} \nu \end{aligned} \tag{9.13}$$

Hence the functions $\cos_n \nu$ and $\sin_n \nu$ both obey the differential equation:

$$s'' + (n-1)s^{n-1}(1-s^n)^{\frac{n-2}{2}} = 0 \tag{9.14}$$

or more generally, the differential equation:

$$\frac{d^2 s}{d\nu^2} + \alpha^2 s^{n-1}(\alpha^n - s^n)^{\frac{n-2}{n}} = 0 \tag{9.15}$$

For each supercircle one can define the perimeter as $2\pi_n$. To determine the half-perimeter π_n we can use the integral value [53]:

$$\pi_n = 2\int_0^1 \frac{dt}{(1-t^n)^{\frac{n-1}{n}}} \tag{9.16}$$

which denotes the quadruple of the area sector between the positive x-axis and the positive y-axis. For even n it is the area enclosed by the whole curve, which yields for the case $n = 2$ the number $\pi_2 = \pi$ of the circle. The calculation of π_n can be traced back to the tabulated factorial [51]. Namely:

$$\pi_n = 2\,\frac{\left(\frac{1}{n}\right)!\left(-\frac{n-1}{n}\right)!}{\left(-\frac{n-2}{n}\right)!} \tag{9.17}$$

n	π_n
2	3.142
3	3.533
4	3.708
5	3.801
6	3.855
7	3.890
8	3.914
9	3.931
10	3.943
∞	4.000

Table 1. See [51].

One finds the values shown in Table 1 and, according to the geometric meaning, one supposes the limit $\pi_\infty = 4$. To confirm this, one introduces the Gamma function which gives:

$$\pi_\infty = 2 \lim_{n\to\infty} \frac{\Gamma(1+\frac{1}{n})\Gamma(\frac{1}{n})}{\Gamma(\frac{2}{n})} = 2\,\Gamma(1) \lim_{x\to 0} \frac{\Gamma(x)}{\Gamma(2x)} \tag{9.18}$$

or since:

$$\Gamma(2x) = \frac{\Gamma(x)\Gamma(x+\frac{1}{2})}{2^{1-2x}\sqrt{\pi}} \quad \text{and} \quad \lim_{x\to 0} \frac{\Gamma(x)}{\Gamma(2x)} = \frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2})} = 2$$

indeed $\pi_\infty = 4$.

Unit supercircles thus have dedicated trigonometric functions [42, 53, 67] and a dedicated value of π_n, defined as the half-perimeter of the super- (or sub-)circle with exponent n (Figure 21 and Table 1) [67].

$$(\cos_n(\vartheta), \sin_n(\vartheta)) = \begin{cases} |x|^n + |y|^n = 1 \\ y = \tan(\vartheta) \cdot x \end{cases} \tag{9.19}$$

For $n = 2$ the functions $\cos_n(\vartheta) = \cos(\vartheta)$ and $\sin_n(\vartheta) = \sin(\vartheta)$, and additionally $\tan_n(\vartheta) = \tan(\vartheta)$. This gives the generalized Pythagorean theorem:

$$(\cos_n(\vartheta))^n + (\sin_n(\vartheta))^n = 1 \tag{9.20}$$

9.4 From Superellipses to Gielis Transformations

Using $\rho = R\cos(\vartheta)$ and $\rho = R\sin(\vartheta)$ and using different exponents n_1, n_2, n_3 gives [7, 40, 92]:

$$\rho(\vartheta; n_1, n_2, n_3) = \frac{R}{\sqrt[n_1]{|\cos(\vartheta)|^{n_2} + |\sin(\vartheta)|^{n_3}}} \tag{9.21}$$

And for the superellipse with semi-major and semi-minor axes A, B:

$$\rho(\vartheta; A, B, n_1, n_2, n_3) = \frac{1}{\sqrt[n_1]{|\frac{1}{A}\cos(\vartheta)|^{n_2} + |\frac{1}{B}\sin(\vartheta)|^{n_3}}} \tag{9.22}$$

The restriction to the Cartesian coordinate system is solved by adding a symmetry parameter $\frac{m}{4}$:

$$\rho(\vartheta; A, B, n_1, n_2, n_3) = \frac{1}{\sqrt[n_1]{|\frac{1}{A}\cos(\frac{m}{4}\vartheta)|^{n_2} + |\frac{1}{B}\sin(\frac{m}{4}\vartheta)|^{n_3}}} \tag{9.23}$$

Examples are shown in Figure 22 (a) to (f) [40]. More generally, Equation (9.23) can be a transformation of any planar function $f(\vartheta)$:

$$\rho(\vartheta; A, B, n_1, n_2, n_3) = \frac{f(\vartheta)}{\sqrt[n_1]{|\frac{1}{A}\cos(\frac{m}{4}\vartheta)|^{n_2} + |\frac{1}{B}\sin(\frac{m}{4}\vartheta)|^{n_3}}} \tag{9.24}$$

Examples of transformations of spirals and trigonometric functions are shown in Figure 22 (g) to (i) and Figure 22 (j) to (l) respectively; see also [46, 56, 90].

The symmetry parameter m can be integer, rational or irrational (Figure 23). Rational values of m generate polygrams. Regular m-polygons are defined as [14]:

$$\rho(\vartheta) = \lim_{n_1 \to \infty} \frac{1}{\left[\left|\cos(\frac{m}{4}\vartheta)\right|^{2(1-n_1 \log_2 \cos\frac{\pi}{m})} + \left|\sin(\frac{m}{4}\vartheta)\right|^{2(1-n_1 \log_2 \cos\frac{\pi}{m})}\right]^{\frac{1}{n_1}}} \tag{9.25}$$

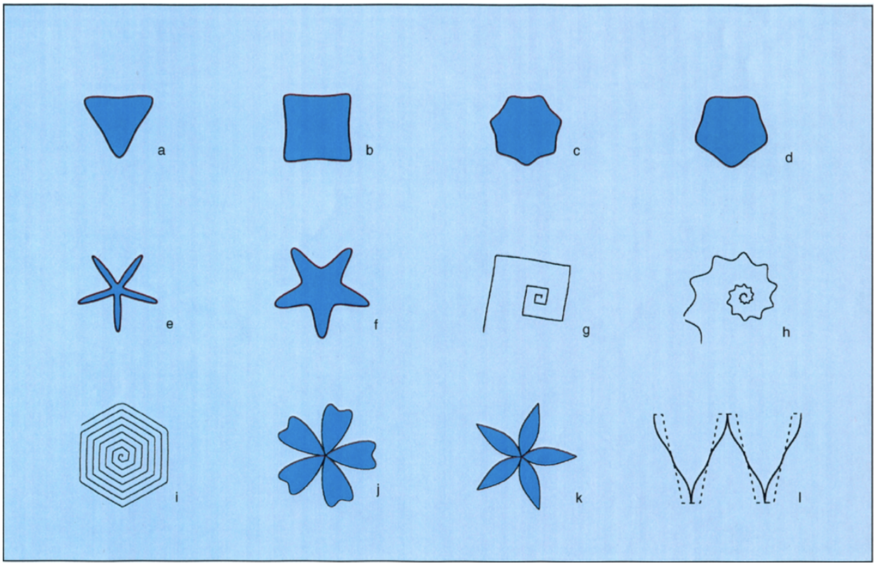

Figure 22. (a)-(d): cross sections of plant stems. (e)-(f): starfish. (g)-(i): transformations of logarithmic (g+h) and Archimedean (i) spirals. (j)-(l): transformations of cosines, as flowers or in wave view [40].

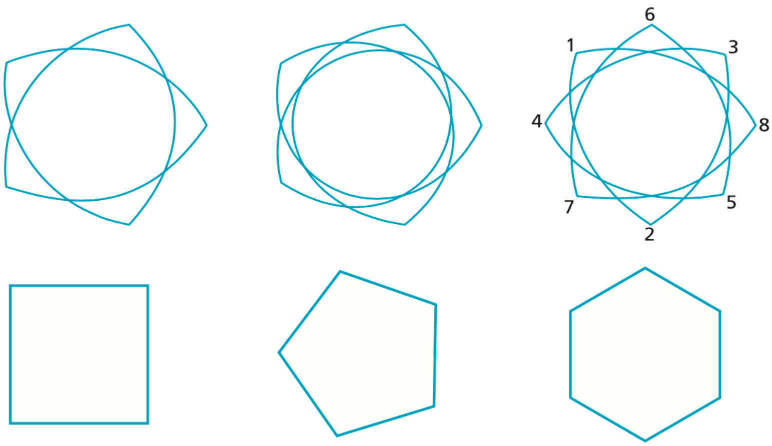

Figure 23. Top row: self-intersecting polygons for $m = \frac{5}{2}, \frac{5}{3}, \frac{8}{3}$. Bottom row: regular polygons for $m = 4, 5, 6$.

Examples are shown in Figure 23 bottom row. Approximations to regular polygons and polygrams are obtained by defining regular Gielis polygons [72]:

$$G_{m,n_{2,3},n_1} = \left\{ \left|\cos(\frac{m}{4}\vartheta)\right|^{n_2} + \left|\sin(\frac{m}{4}\vartheta)\right|^{n_3} \right\}^{-\frac{1}{n_1}} \quad \text{with} \quad n_{2,3} = \frac{m^2}{16} \cdot n_1 \tag{9.26}$$

For $m > 5$, the deviations of $G_{m,n_{2,3},n_1}$ from true regular polygons are less than 1%, monotonically decreasing for increasing m [72].

$$G_{m,n_{2,3},n_1} = \left\{ \left|\cos(\frac{m}{4}\vartheta)\right|^{n_2} + \left|\sin(\frac{m}{4}\vartheta)\right|^{n_3} \right\}^{-\frac{1}{n_1}}$$

Using Equation (9.23) as equality:

$$\rho(\vartheta) = \frac{1}{\sqrt[n_1]{|\frac{1}{A}\cos(\frac{m}{4}\vartheta)|^{n_2} + |\frac{1}{B}\sin(\frac{m}{4}\vartheta)|^{n_3}}}$$

defines the curve as a boundary of a disk.

The inequality:

$$\rho(\vartheta) \leq \frac{1}{\sqrt[n_1]{|\frac{1}{A}\cos(\frac{m}{4}\vartheta)|^{n_2} + |\frac{1}{B}\sin(\frac{m}{4}\vartheta)|^{n_3}}}$$

defines boundary and disk and the inequality $\rho(\vartheta) >$ defines all space outside this domain. Using ranges one can define annuli as well (Figure 24).

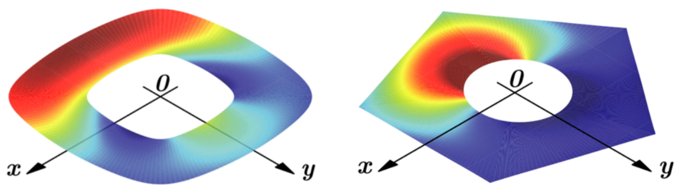

Figure 24. Annuli with same or different outside and inside boundaries. On the right the outer boundary is a regular pentagon. [14]

9.5 Generalizations

R. Chacón proposed the following equation [25]:

$$\rho(\vartheta) = \left\{ \left|\cos(\Phi_{\mathrm{I,II}}\vartheta)\right|^{n_2} + \left|\sin(\Psi_{\mathrm{I,II}}\vartheta)\right|^{n_3} \right\}^{-\frac{1}{n_1}} \tag{9.27}$$

where $\Phi_{\mathrm{I,II}}$ and $\Psi_{\mathrm{I,II}}$ are suitable amplitudes of Jacobi elliptic functions. The motivation of this choice is because the solution of many nonlinear physical models are expressed in terms of such functions [25]. Dealing with modeling of natural shapes this selection appears quite restrictive, since the non-linearity is restricted by the use of these particular functions. This leads to the following generalization [48]:

$$\rho(\vartheta) = c(\vartheta) \left[\left| \frac{1}{A} \cos(\frac{m_1 f_1(\vartheta)}{4}) \right|^{n_2} + \left| \frac{1}{B} \sin(\frac{m_2 f_2(\vartheta)}{4}) \right|^{n_3} \right]^{-\frac{1}{n_1}} \tag{9.28}$$

Here $\vartheta \in [-\pi, \pi]$; f_1, f_2 and $c(\vartheta)$ are continuous functions; m_1, m_2, A, B, $n_{1,2,3}$ are real numbers; and A, B, n_1 are not zero. Division by 4 in the preceding formula is unnecessary of course, since the same results can be simply obtained by changing the values m_1 and m_2. Actually, this division is only assumed in order to preserve the original form of the Gielis Superformula and the special case of Lamé curves for $m = 4$. It is then possible to impose conditions, such as conditions for functions to be increasing $f(-\pi) = -\pi$, $f(\pi) = \pi$, closed $f(-\pi) = f(\pi)$, or to pass through the origin [48]. Some further examples are shown in Figure 25.

9.6 Supercircles and Superparabolas

The structural form of the above equations is both Pythagorean-compact and topologically simple. The basic structure is [41]:

$$\rho(\vartheta) = \frac{1}{\sqrt[\ddots]{|\cos(::)|^{\cdots} \pm |\sin(::)|^{\vdots}}} \tag{9.29a}$$

or

$$\rho(\vartheta) = \frac{1}{\sqrt[\ddots]{|f_1(::)|^{\cdots} \pm |f_2(::)|^{\vdots}}} \tag{9.29b}$$

and the variables and exponents $(::, \cdots, \vdots, \ddots)$ can either be numbers or functions for evolution along a time or space axis or both. When all exponents $(\cdots, \vdots, \ddots) = 2$, Equation (9.29a) describes the circle, hence the name Pythagorean-compact. If we wish to describe the complexity of an object, it is not only the degree, but also the number of monomials in the polynomials, which describe the curve or object. In the case of Lamé curves, irrespective of the degree, the number of monomials is one in each variable. This is precisely the case for Lamé-Gielis curves (including those in Figure 25): not only are they Pythagorean-compact, but also topologically simple [41]. Addition of monomials x^n, y^m results in Lamé curves, but the operation multiplication or division can also be used. Allometric laws $x^n = ky^m$ result from multiplication or division of the monomials, but geometrically these are superparabolas (or superhyperbolas) (Table 2).

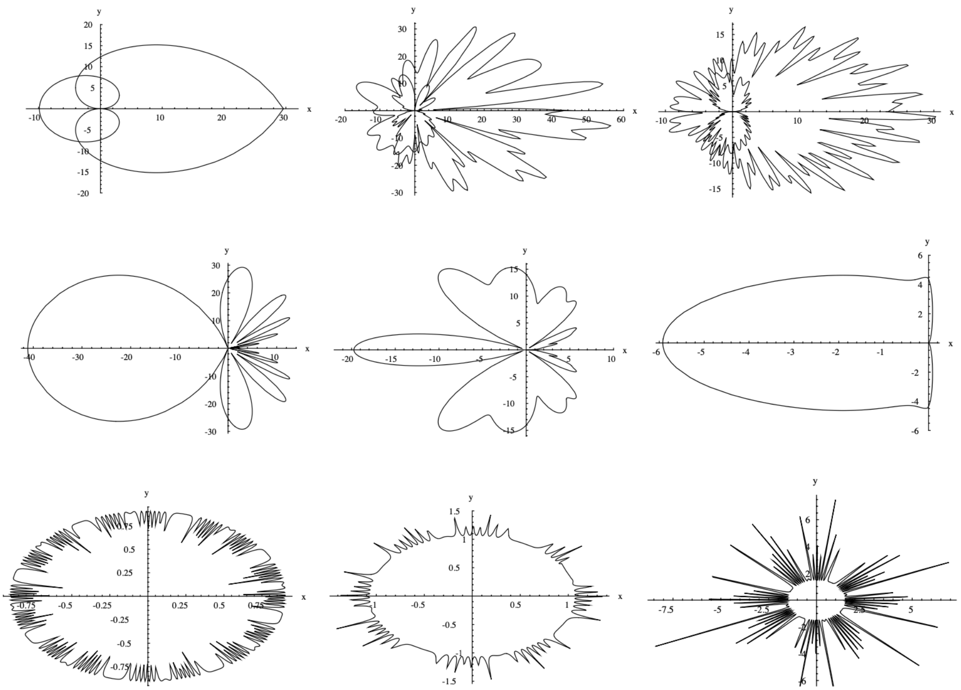

Figure 25. Examples of generalized Gielis transformations [48].

Variables	Planar curves	Types	Special means
$x^n + y^n$	Supercircles & Superellipses	Lamé curves	Arithmetic mean $\frac{x+y}{2}$ (for $n = 1$)
$x^n \cdot y^m$	Superparabolas & Superhyperbolas	Power laws	Square of geometric mean $\sqrt{x \cdot y}$ (for $n = m = 1$)

Table 2. Lamé curves and power laws as generalizations of conic sections [46].

In the same way as supercircles are generalizations of circles, these power relations are superparabolas which are generalizations of the classic parabola $y = x^2$. In Figure 26 left, superparabolas $y = x^{n/m}$ are shown in the interval $[0; 1]$ and the exponents range from $n = 1/2$ to $n = 2$ with steps of $1/5$. The cases for $n > 1$ and $n < 1$ have $y = x$ with $n = 1$ as the symmetry axis (the bisectrix). The classic parabola is a machine that turns a rectangle with area $y \cdot 1$ into a square with the same area and side x, which is the geometric mean. In the same way, a superparabola $y = x^{n/m}$ turns a beam with an n-volume into a cube with an m-volume (for $n < m$) [41].

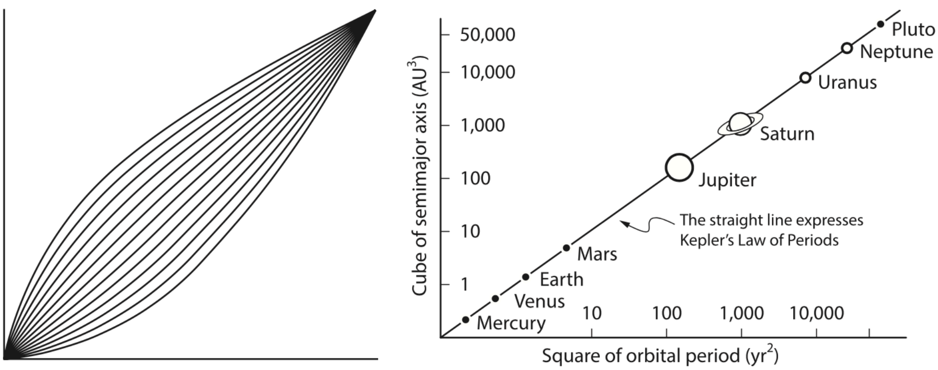

Figure 26. Left: sub- and superparabolas in the interval [0; 1]. Right: Kepler's Law [41].

The trigonometry of the parabola yields interesting insights. The value of the half-perimeter π_{par} is a rational number and is directly related to Archimedes' *Quadrature of the Parabola*. Moreover, the values of the associated $\sec_{par}$ and $\cos_{par}$ at 45° give the Golden Ratio φ and its inverse $1/\varphi$ respectively [32, 33, 91]. The generalization of these trigonometric functions to super- and subparabola, analogous to the trigonometry on supercircles, needs to be developed.

In physics, biology and economy, power laws are found everywhere, from the size of cities to the power noise in time series. The Cobb-Douglas production function $V = \gamma K^{\delta} L^{1-\delta}$ is one example from economics, with the output of a Process V defined by Capital K and Labor L, which can be substituted to a certain extent depending on the substitution parameter δ. This is equivalent to an expression like $z = x^n \cdot y^m$. In the case of Cobb-Douglas $n = 1-m$. Actually, the Cobb-Douglas model is a limiting case of the CES (constant elasticity of substitution) production models (in the case of $\rho = 0$ elasticity reduces to unity) [3]:

$$V = \gamma \left[\delta K^{-\rho} + (1-\delta) L^{-\rho}\right]^{-\frac{1}{\rho}} \tag{9.30}$$

with K = capital, L = labor, γ an efficiency parameter, ρ a substitution parameter (transform of elasticity of substitution) and δ a distribution parameter (δ and $1-\delta$ make this into a weighted mean). The generalized form of CES production functions with an arbitrary number of inputs is [29]:

$$F(x_1, x_2, \ldots, x_n) = A \left(\sum_{i=1}^{n} a_i^{\rho} x_i^{\rho} \right)^{\frac{\gamma}{\rho}} \quad \text{with} \quad a_i, \gamma, A, \rho \neq 0 \text{ and } \rho < 1$$