Chapter 4. Fourier and the Trigonometric Series

The education of Jean-Baptiste Joseph Fourier [5] (Figure 5) was first carried out by the Benedictines and later in a military school. He would have liked to pursue a military career, but was not accepted due to his modest origins.

In the spring of 1793 he began to pursue political ideals. Favorable to the ideas of the Revolution, he joined the Société Populaire of Auxerre. As a consequence of his courageous defense of some victims of The Terror, he was arrested on July 4th, 1794 and risked to be guillotined, but luckily was released on the 28th following the fall of Maximilien Robespierre. He then entered the École Normale Supérieure, where he had as professors, among others, the leading scientists of that time: Joseph-Louis Lagrange, Pierre-Simon Laplace and Gaspard Monge for Mathematics and Claude Louis Berthollet for Chemistry. In September 1794, he was again arrested on charges of having been a follower of Robespierre, but was released probably due to intervention

Figure 5. Jean-Baptiste Joseph Fourier.

by his professors or by Napoleon himself. From 1795 he was assistant to Lagrange at the École Centrale des Travaux Publiques, later baptized École Polytechnique, a military academy established by Monge, and in 1797 he succeeded Lagrange in the role of Professor of Analysis and Mechanics. He was said to be an excellent teacher.

In 1798, together with 164 other scholars including Monge and Berthollet (the so-called *Légion de Culture*), Fourier sailed from Toulon with Napoleon and general Kléber to Egypt, where he computed the height of the pyramids of Memphis by measuring the height of each step and using the least squares method in order to minimize the errors. Following the British victories, all the members of the Institute returned back to France in 1801. Fourier was then appointed by Napoleon, once again on the recommendation of Monge and Berthollet, as Prefect of the Department of Isère which had Grenoble as its capital. It was in Grenoble that he studied the propagation of heat, modeling the evolution of temperature by means of trigonometric series.

Fourier studied the trigonometric series [100] using the properties of orthogonality:

$$\int_{-\pi}^{\pi} \cos(hx)\cos(kx)dx = \begin{cases} 2\pi & if \quad h=k=0 \\ \pi & if \quad h=k\neq 0 \\ 0 & if \quad h\neq k \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(hx)\sin(kx)dx = \begin{cases} \pi & if \quad h=k\neq 0 \\ 0 & if \quad h\neq k \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(hx)\cos(kx)dx = 0 \qquad \forall h,k$$

and stated that any periodic function in $[-\pi,\pi]$ could have an expansion into a convergent series of sines and cosines. In Figures 6 and 7 examples of expansions are shown.

The study of the trigonometric series [100, 106] was undertaken by Fourier in the context of the problem of heat transmission. It is said that Fourier wanted to establish at what depth a cellar should

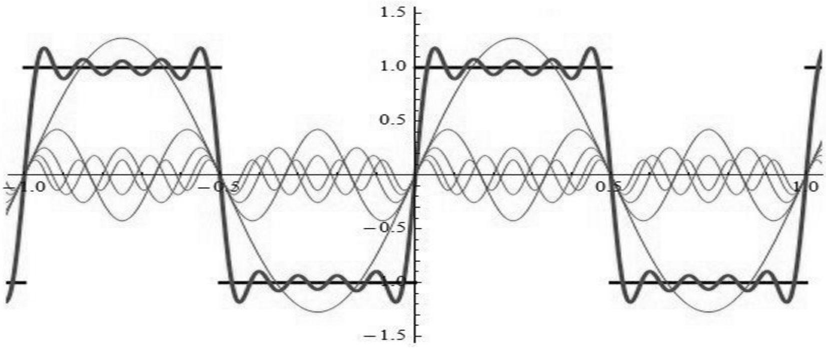

Figure 6. Approximation of a square wave.

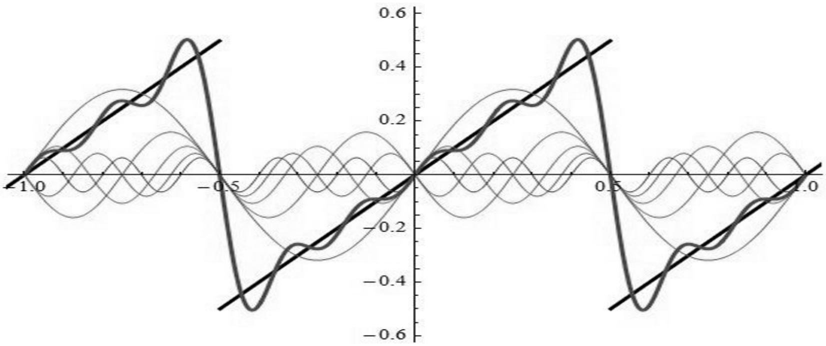

Figure 7. Approximation of a sawtooth wave.

be built to keep a constant temperature throughout the year (an important condition for the conservation of wine). The problem of heat propagation became the topic of the Grand Prix of the Académie des Sciences Mathématiques in 1811. Fourier presented his work for the purpose of winning the prize. The Commission, composed of Lagrange (Figure 8 left), Laplace (Figure 8 right), Malus, Haüy and Lacroix rewarded Fourier. However, Lagrange criticized the work in terms of rigor and difficulty in generalizing the results. Therefore, the work was not published in the Mémoires de l'Académie.

It was only in 1822 that Fourier succeeded in publishing *La Théorie Analytique de la Chaleur*. It became one of the classics of mathematics, including part of his work from 1812.

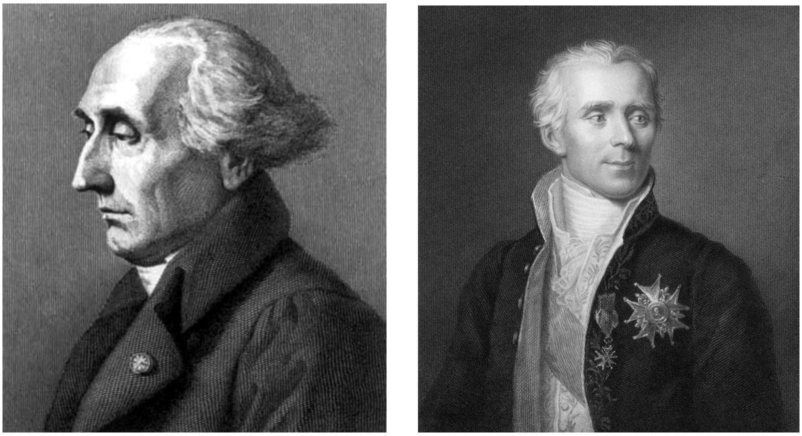

Figure 8. Left: Joseph-Louis Lagrange. Right: Pierre-Simon Laplace.

But before that many things had to happen, which involved Fourier in the stormy periods of Napoleon's fall. In 1814 Napoleon abdicated and Louis XVIII took the throne. It was a difficult time for Fourier, but his diplomatic ability managed to save his position in Grenoble. In the period of Napoleon's return and after the battle of Waterloo he had to navigate between the interests of the Empire and those of the Restoration. He was first set aside and later, in 1817, he was finally admitted to the Académie. In 1822, he was appointed Secrétaire Perpétuel of the Académie for Mathematical Sciences. After the death of Lagrange and Laplace, Fourier had moments of strong rivalry with both Poisson and Cauchy, but he had the support of his devoted friends Sturm, Navier, Dirichlet and Liouville. In fact, as his results were not published yet, they were used by colleagues without mentioning it.

Among other contributions of Fourier it is worth to mention the solution of a system of linear inequalities, a method for finding a constrained minimum in $\mathbf{R}^n$ that prefigures the simplex algorithm and his article *Remarques Générales sur les Températures du Globe Terrestre et des Espaces Planétaires* in which, studying the effects of the presence of the atmosphere on solar radiation, he conjectured what is now called the *greenhouse effect.*

4.1 On the Convergence of Trigonometric Series

The Fourier system is shown to be *complete*. The systems of the so-called classical orthogonal polynomials (Jacobi, Gegenbauer, Legendre, Chebyshev, etc.) are also complete [60]. The definition of complete system requires that every function can be approximated in norm, less than a fixed ε, by means of a finite linear combination of elements of the system. Therefore, if for example we took only the *sine* functions, which are odd, there would be no possibility of approximating the functions that were not themselves odd. Similarly for the *cosine* functions. Therefore, in the case examined by Fourier it is essential to include both the sine and cosine functions in the basis.

The problem of the convergence of Fourier series is very complex and has only recently been solved by Lennart Carleson (Figure 9), winner of the 2006 Abel Prize. In fact, three types of convergence are considered:

1. Convergence in quadratic mean
2. Pointwise convergence
3. Uniform convergence

Referring for shortness to the expansions with respect to a complete system of polynomial functions with unit weight $w(x) \equiv 1$, denoting by

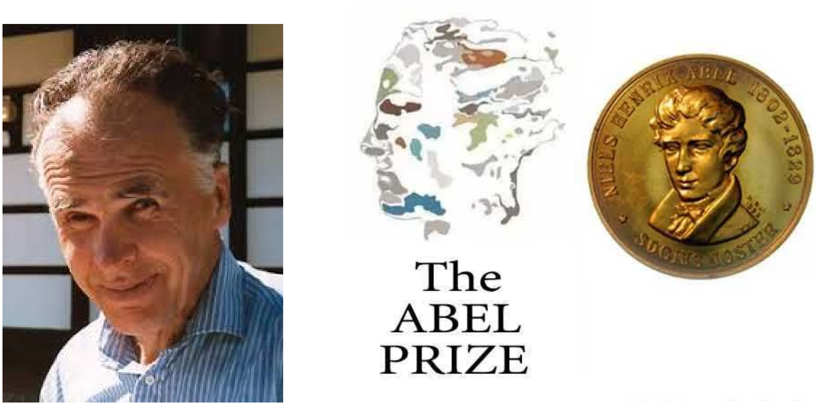

Figure 9. Lennart Carleson was the winner of the 2006 Abel Prize.

$\sum_{k=0}^{n} f_k u_k(x)$ the partial sum of the Fourier series, with coefficients as indicated above, we must have respectively:

1. $\int_a^b \left|\sum_{k=0}^{n} f_k u_k(x) - f(x)\right|^2 dx \to 0$
2. If $x \in (a,b) \Rightarrow \sum_{k=0}^{n} f_k u_k(x) \to f(x)$
3. If $x \in [\alpha,\beta] \subset (a,b) \Rightarrow \max_{x\in[\alpha,\beta]} \left|\sum_{k=0}^{n} f_k u_k(x) - f(x)\right| \to 0$

In the case of a complete system, there is always the convergence in quadratic mean. Pointwise convergence occurs only in the points of continuity of the function in which further there exist both the right derivative $f'(x)^+$ and the left one $f'(x)^-$ (in particular where the function is differentiable). In the points where the function has a discontinuity of the first kind (right limit $f(x)^+$ and left limit $f(x)^-$ bounded) and again $f'(x)^+$ and $f'(x)^-$ exist, we have the convergence towards the midpoint:

$$\frac{f(x)^+ + f(x)^-}{2}$$

Uniform convergence can occur only in bounded and closed intervals contained in the intervals of continuity of the function.

Lennart Carleson proved that if the integral of the square of the function in (a,b) is bounded, then the pointwise convergence holds up to a set of points of measure zero in (a,b).